Periods, arithmetic & the Hodge structure for Calabi-Yau manifolds

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geometry of families of Calabi-Yau varieties and that of their moduli spaces

arithmetic of families of Calabi-Yau varieties and modularity properties

PERIODS

physics

- black hole solutions in string theory
- scattering amplitudes

etc, etc, ...

Collaborators

- P. Candelas & F Rodríguez-Villegas 2000, 2004
- P. Candelas, D. van Straten & M Elmi 2019
- P. Candelas, D. van Straten 2021
- P. Candelas, J McGovern & P Kuusela 2021-2024
- P. Candelas & P Kuusela 2024
- P. Candelas, D van Straten, N. Gegelia in progress 2025
- P. Candelas & E Svanberg in progress 2025.

1. Calabi-Yau varieties (very brief)

2. Calabi-Yau varieties OVER FINITE FIELDS

   mostly a classical discussion

   2.1 Zeta funtions

   2.2 Exercise: counting points mod p

   2.3 WHY? intuitive notion only

   2.4 Back to the zeta function

3. The attractor mechanism

4. Arithmetic of attractor varieties

5. Conclusions

1

CALABI-YAU VARIETIES

Mathematical objects of interest:

$\underbrace{\text{algebraic varieties}}_{\text{set of solutions of}}$ with certain special properties

$\boxed{P(\varphi, \underline{x}) = 0, \quad \underline{x} \in \mathbb{A}}$

Calabi-Yau
· Kähler · $c_1 = 0$

polynomials with complex coefficients φ

This talk: concerned with $d=3$

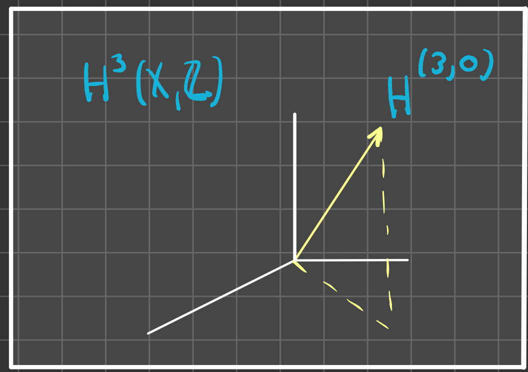
▸ It is a theorem that $\exists!$ (up to a constant) a nowhere vanishing $(3,0)$-form Ω which is holomorphic $(d\Omega = 0)$

so $\quad h^{(3,0)} = \dim H^{(3,0)} = \dim H^{(0,3)} = 1$

- $H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$

  Hodge structure

  $\dim H^3 = b_3 = 1 + h^{(2,1)} + h^{(1,2)} + 1 = 2(1 + h^{(2,1)})$



- Vary the complex structure

  $\longrightarrow$ Ω moves inside $H^3(X,\mathbb{Z})$ (variations of the HS)

($h^{1,2}$ = $\mathbb{C}$-dim moduli space)

Examples: very many!

- $\mathbb{P}^4[5]$ eg $\sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 = 0$, $(x_1, \ldots, x_5) \in \mathbb{P}^4$

  $c_1 = 0$

- We have in mind a particular example:

  Verrill 1996, Hulek & Verrill 2005

(quotient of the)

Mirror of a CICY

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array}\right]$$

$h^{11} = 5$

quotient

$h^{11} = 1$

why this example? exhibits interesting arithmetic properties which have an interpretation in BH solutions of string theory (also in amplitudes)

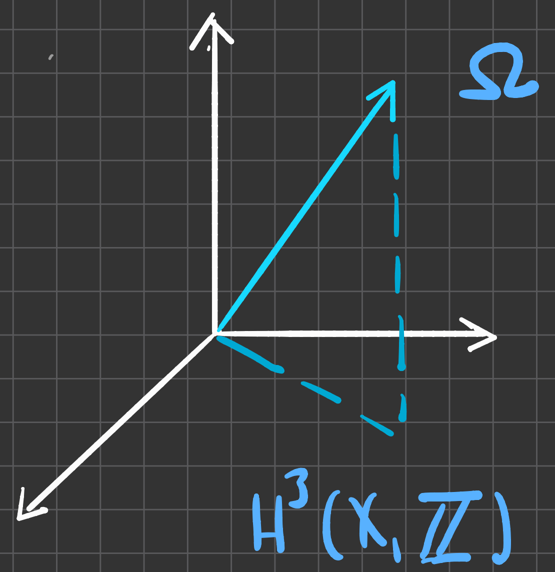
Periods and the complex structure

↳ there is a canonical way to give coordinates on the space of complex structures

Recall: Ω is defined up to a scale but is otherwise unique $\Rightarrow$ defines a line in H^3

- study the variations of the complex structure by studying how Ω varies in H^3

- the coordinates of this line are the periods (which then vary as we vary the complex structure)



That is, $\Omega(\varphi) = \varpi^i \alpha_i$ ← basis of $H_3(X,\mathbb{Z})$

Bryant & Griffiths: the complex structure is completely determined by (half of) the periods

Periods determine $\underbrace{\text{the geometry of the moduli space}}_{\text{special geometry}}$ ⇝ physics!

↑ also have arithmetic content

Periods are calculable: they satisfy a differential eq of degree b_3 (Picard-Fuchs equation)

$$\mathcal{L}\varpi = 0$$

Focus today 1-parameter examples ($b_3 = 2(1+h_{21}) = 4$)

- There is a prescription to obtain the PF eq

(Dwork and Griffiths; Gelfand, Kapranov and Zelevinskii)

- One parameter families of CY : $b_3 = 4$

$\deg \mathcal{L} = 4$

Solutions around MUM point $\varphi = 0$

$(T-1)^4 = 0, \; (T-1)^3 \neq 0$

Frobenius basis

$$\varpi_0(\varphi) = \sum_{m=0} a_m \varphi^m = f_0(\varphi) \longleftarrow \text{fundamental period}$$

$$\varpi_1(\varphi) = f_0(\varphi) \log \varphi + f_1(\varphi)$$

$$\varpi_2(\varphi) = f_0(\varphi) \log^2 \varphi + 2 f_1(\varphi) \log(\varphi) + f_2(\varphi)$$

$$\varpi_3(\varphi) = f_0(\varphi) \log^3 \varphi + 3 f_1(\varphi) \log^2(\varphi) + 3 f_2(\varphi) \log \varphi + f_3(\varphi)$$

f-series are regular

2 CALABI-YAU VARIETIES OVER FINITE FIELDS

2.1 Zeta functions

Let X_φ be a family of algebraic varieties st X_φ is a CY hypersurface with defining polynomial $P(\underline{x}, \varphi)$

Let $\varphi \in \mathbb{Q}$

Questions:

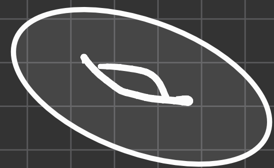
- how many solutions of $P(\underline{x}, \varphi) = 0$ are there over $\mathbb{Q}$ (ie $x_i \in \mathbb{Q}$)
- how does this number vary with φ?

¡ TOO HARD !

For elliptic curves for example :

" Millenium prize problem "

Birch & Swinnerton-Dyer conjecture



1dim CY

One learns a lot however by "reducing mod p" where p is a prime number, that is, by working over finite fields $\mathbb{F}_{p^k}$, $k = 1, 2, \ldots$

$(\mathbb{F}, +, \times)$

field with p^k elements

[simplest: $\mathbb{F}_p \longrightarrow$ integers mod p

eg $\mathbb{F}_7$ $\quad x: 0, 1, 2, 3, 4, 5, 6$

$x^{-1}: *, 1, 4, 5, 2, 3, 6$

[$p \neq$ prime does not work ! $\mathbb{Z}$-mod 6 $\quad 2 \cdot 3 \equiv 0 \bmod 6$]

$\mathbb{F}_{p^2} \longrightarrow \mathbb{F}_p[\alpha] = \{ a + \sqrt{\alpha}\, b,\ a, b \in \mathbb{F}_p,\ \alpha \neq \text{a square in } \mathbb{F}_p \}$

eg 3, 5 not squares in $\mathbb{F}_7$

$\mathbb{F}_7$ $\quad x^2: 0, 1, 4, 2, 2, 4, 1$

So let $\varphi \in \mathbb{F}_p$:

The fundamental quantities of interest are

$N_k(\varphi)$ = number of solutions of $P(\underline{x},\varphi)=0$ over $\mathbb{F}_{p^k}$

Generating function $\rightsquigarrow$ zeta function

$$\zeta_X(T,p;\varphi) = \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k} N_k(\varphi)\, T^k \right\}$$

depends on both p & φ

properties $\rightsquigarrow$ Weil conjectures

(proven by Dwork, Deligne, Grothendieck)

In particular,

$\zeta(T)$ is a rational function (Dwork)

(true even if the variety is singular)

Moreover: (for smooth varieties)

$$\zeta(T) = \frac{R_1(T)\, R_3(T) \cdots R_{2d-1}(T)}{R_0(T)\, R_2(T) \cdots R_{2d}(T)}$$

$R_i(T)$ polynomials with integer coeffs

$\deg R_i = b_i$ i-th Betti number

$R_0(T) = 1 - T$, $R_{2d}(T) = (1 - p^d T)$

"simple" case: X is a point.

$$N_k = 1 \quad \forall k$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{k} N_n T^k = \sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1-T)$$

$$\Rightarrow \zeta_{\text{point}}(T) = \frac{1}{1-T}$$

Remark: $$\prod_p \zeta_{\text{point}}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta_{\mathbb{R}}(s)$$

$\boxed{\text{enter L-series}}$

Elliptic curves

CY 1 dim

$\mathcal{E}$ over $\mathbb{Q}$

$$\zeta(T) = \frac{1 + a_p T + p T^2}{(1-T)(1-pT)}$$

$a_p = p+1-N_p$

$p \neq$ prime of bad reduction

$\mathcal{E}$ is associated to a modular form

Taniyama-Shimura modularity conjecture

- proof : Wiles, Breuil, Conrad, Diamond, Taylor

(the proof of the Fermat's conjecture follows from this)

L-function : $\text{''}\prod_p \zeta(p^{-s})\text{''} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

$a_n \rightsquigarrow$ n-th coeffient of a $w=2$ modular form

constructed from the a_p

of $\Gamma_0(N) < SL(2,\mathbb{Z})$

$\Gamma_0(N) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ st $c = 0 \mod N$

conductor : its prime factors $\rightsquigarrow$ primes of bad reduction

modular form : $f(\tau) = \sum_n a_n q^n$, $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$

- holomorphic
- for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$: $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ k = weight

CY threefolds

Hodge numbers

$h^{i,j} = h^{j,i}$

$h^{1,1} = h^{2,2}$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \circ & & \circ & & \\
 & \circ & & h^{11} & & \circ & \\
1 & & h^{21} & & h^{12} & & 1 \\
 & \circ & & h^{22} & & \circ & \\
 & & \circ & & \circ & & \\
 & & & 1 & & &
\end{array}$$

$b_0 = 1$

$b_1 = 0$

$b_2 = h^{11}$

$b_3 = 2(1 + h^{21})$

$b_4 = b_2 = h^{11}$

$b_5 = 0$

$b_6 = 1$

$\deg R(T) = b_3$

$$\zeta_X(T) = \frac{\cancel{R_1(T)} \quad R(T) \quad \cancel{R_5(T)}}{(1-T)\underbrace{(1-pT)^{h^{11}}\,(1-p^2T)^{h^{11}}}\,(1-p^3T)}$$

smooth CY

(Picard group generated by divisors defined over the ground field $\mathbb{F}_p$)

$b_2 = b_4 = h^{11}$

1-parameter families ($h^{2,1}=1$) of CY varieties

$b_3 = 1+1+1+1 = 4$ so $\deg R_3 = 4$

$R(T) = 1 + a_p T + b_p\, p\, T^2 + a_p p^3 T^3 + p^6 T^3$

where $N_p = p^3 + h^{11} p^2 + h^{11} p + 1 + a_p$

$N_{p^2} = p^6 + h^{11} p^4 + h^{11} p^2 + 1 + 2p b_p - a_p^2$

- How do we compute a_p, b_p? How do these vary with φ?
- And how is this story related to the periods?
- Is there an analogue of the modularity theorem for $E/\mathbb{Q}$?
- $\deg R_3 < 4$ @ singularities

2.2 AN EXERCISE: COUNTING POINTS MOD P

- Fermat's little theorem $\quad a^p \equiv a$

$$\Rightarrow \quad a\,(a^{p-1}-1) \equiv 0 \quad \Rightarrow \quad a^{p-1} \equiv \begin{cases} 1 & a \neq 0 \\ 0 & a = 0 \end{cases}$$

Then:
$$1 - P(\underline{x},\varphi)^{p-1} \equiv \begin{cases} 1, & \forall\, \underline{x} \text{ st } P(\underline{x},\varphi) = 0 \\ 0, & \forall\, \underline{x} \text{ st } P(\underline{x},\varphi) \neq 0 \end{cases}$$

$$\therefore \quad \nu_p(\varphi) \equiv \sum_{\underline{x} \in \mathbb{F}_p^n} \left(1 - P(\underline{x},\varphi)^{p-1}\right)$$

n-varibles

Chevalley, Warning 1935 ..., P. Candelas, XD, F Rodriguez-Villegas 2000, 2004

... E. Panzer, F Brown

(mirror) quintic:

$$\nu_p(\varphi) \equiv \sum_{m=0}^{[p/5]} a_m \varphi^m \equiv {}^{[p/5]}\varpi_0(\varphi)$$

truncated fundamental period.

- result extends to all hypersurfaces in toric varieties and $h^{21} \geq 1$
- why?
- Want to count points exactly!

Counting points exactly :

In progress with P Candelas & E Svanberg :

primitive part of N_p

truncated f-series

$$= \sum_{\underline{\ell}, \underline{j} = \underline{0}}^{\infty} \frac{h_{\underline{\ell}, \underline{j}}}{\underline{\ell}!} \left(\frac{p}{1-p} \underline{\vartheta} \right)^{\underline{\ell}+\underline{j}} \; {}^{[p-2]}f^{\underline{\ell}}(\varphi)$$

$\underline{\ell}, \underline{j} = (\ell_1, \dots, \ell_n)$ $\qquad \underline{\vartheta} = \vartheta_1 \vartheta_2 \cdots \vartheta_n$ $\qquad \vartheta_i = \varphi_i \dfrac{\partial}{\partial \varphi_i}$

mirror quintic

$$= {}^{[p-2]}f_0(\varphi) + \left(\frac{p}{1-p}\right) f_1'(\varphi) + \frac{1}{2} \left(\frac{p}{1-p}\right)^2 f_2''(\varphi) \cdots$$

[F Beukers & M Vlasenko ; M Kerr

K3 : ... R Davis, J Duclos, E Orbis, T Ribeiro, A Salerno, L Sturman, U Whitcher ; T Kelly]

2.3 WHY? intuitive notion only

Frobenius :

Let X be an algebraic variety over $\mathbb{F}_p$ defined by the zero locus of a polynomial

$$P(\underline{x}) = \sum c_{\underline{m}} \underline{x}^{\underline{m}}$$

$c_{\underline{m}} \in \mathbb{F}_p$

$\underline{x} = \{x_1, \dots, x_n\}, \ x_i \in \overline{\mathbb{F}_p}$

$\underline{x}^{\underline{m}} = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$

Suppose $\underline{x} \longrightarrow \underline{x}^p$ $\boxed{\text{Frobenious map}}$

Then: $P(\underline{x}) = 0 \iff P(\underline{x}^p) = 0$

$$P(\underline{x})^p = 0 = \sum_{\underline{m}} c_{\underline{m}}^p \, \underline{X}^{p\underline{m}} = \sum_{\underline{m}} c_{\underline{m}} \, \underline{X}^{p\underline{m}}$$

$(a+b)^p = a^p + b^p$

ie $\underline{x}^p$ satisfies the same equation as $\underline{x}$
and the Frobenious map is an automorphism

Fixed points $(\underline{x}^p = \underline{x})$ of the Frobenious $\longleftrightarrow$ precisely the points over $\mathbb{F}_p$

"p-adic" version of the Lefschetz fixed point theorem from topology

$$N \text{ "=" } \sum_{k \geq 0} (-1)^k \operatorname{Tr}(f_* \mid H_k(X))$$

trace over the Frobenius action on the cohomology of X

linear map induced by the Frobenius on the homology of X

$H^k(X, \mathbb{Q}_p)$ p-adic cohomology

2.4

Back to the zeta function

1-parameter families: $h^{2,1}=1$, $b_3 = 1+1+1+1 = 4$ so $\deg R_3 = 4$

$$\zeta_X(T) = \frac{R(T)}{(1-T)(1-pT)^{h^{11}}(1-p^2T)^{h^{11}}(1-p^3T)}$$ smooth CY

$$R(T) = 1 + a_p T + b_p\, p\, T^2 + a_p p^3 T^3 + p^6 T^3$$
$$= \det(\mathbb{1} - T\,\mathcal{U}(\varphi))$$ (eg $a_p = -\mathrm{tr}\,\mathcal{U}$, ...)

$\mathcal{U} = \mathrm{Frob}_3^{-1} : H^3(X) \longrightarrow H^3(X)$ map induced by the Frob automorphism on H^3

4x4 matrix constructed from the periods of Ω

singular CY: $\deg R(T) < 4$

Need to compute $a_p, b_p, \ldots$

Counting points exactly

* $\boxed{\text{$\zeta_\varphi(T)$ can be "quickly" computed for $\varphi \in \mathbb{F}_p$ $(\varphi = 0, 1, \ldots, p-1)$ for many primes p}}$

P Candelas, XD & D van Straten (2021) — 1 parameter families

P Candelas, XD, P Kuusela (2024) — multi-parameter families

based on the deformation method developed first by Dwork & Lauder

To appear soon: P Kuusela, M. Atwood, M Mosso Rojas, M Stepniczka

Many questions arise: certainly ready for substancial experimentation.

▸ One can construct L-functions

1 parameter families

$$L(s,\varphi) = \text{"}\prod_p \underbrace{R(p^{-s},\varphi)}_{\text{spinor zeta function } w=3}{}^{-1}\text{"}$$

What are the properties of this L-function?

- functional equation
- formula for the conductor
- modularity $\longleftrightarrow$ paramodular forms of $\Gamma(N) \subseteq Sp(4)$ for 1-parameter cases

Langlands programme

[G Tornaria, A Pacetti, J. Voight, V. Golyshev ....

P Candelas, XD, N Gegelia, D van Straten, in progress]

Again at this time these are very **hard** questions

modularity is <u>not</u> classical modularity except in some special cases

➤ rigid CY $(h^{2,1}=0)$ $\Rightarrow$ $\deg R = 2$

$$R(T) = 1 - a_p T + p^3 T^2$$

F. Gouvea + N. Yui (2009), Dieulefait + Manoharmayum (2003)

rigid CYs over $\mathbb{Q}$ are modular

$a_p \rightsquigarrow$ p-th coeff in q-expansion of a modular form of some $\Gamma_0(N)$, $w=4$

▶ What happens at singularities?

necessary to properly understand the L-function

eg conifold singularities

$$R(T) = (1-\epsilon_p T)(1 - a_p T + p^3 T^2)$$

(for one parameter families)

a_p = p-th coeff in q-expansion of the eigenform g of weight 4 of $\Gamma_0(N)$

mirror quintic: $\varphi = 1/5$ $N = 25$ (Schoen)

HV family: $\varphi = 1, 1/9, 1/25$ (not hypergeometric)

Recap :

While at this time it is hard to say in general what the properties of a_p & b_p are (and there are many conjectures)

↳ interesting things happen for special values of φ

eg · HS of CY CYs ⟺ RCFT H Jockers, P Kuusela, M Sarve 2025

· BH physics ⟺ decomposition of the HS

③ THE ATTRACTOR MECHANISM

(Ferrara, Kallosh 95, ... Greg Moore 98 ----)

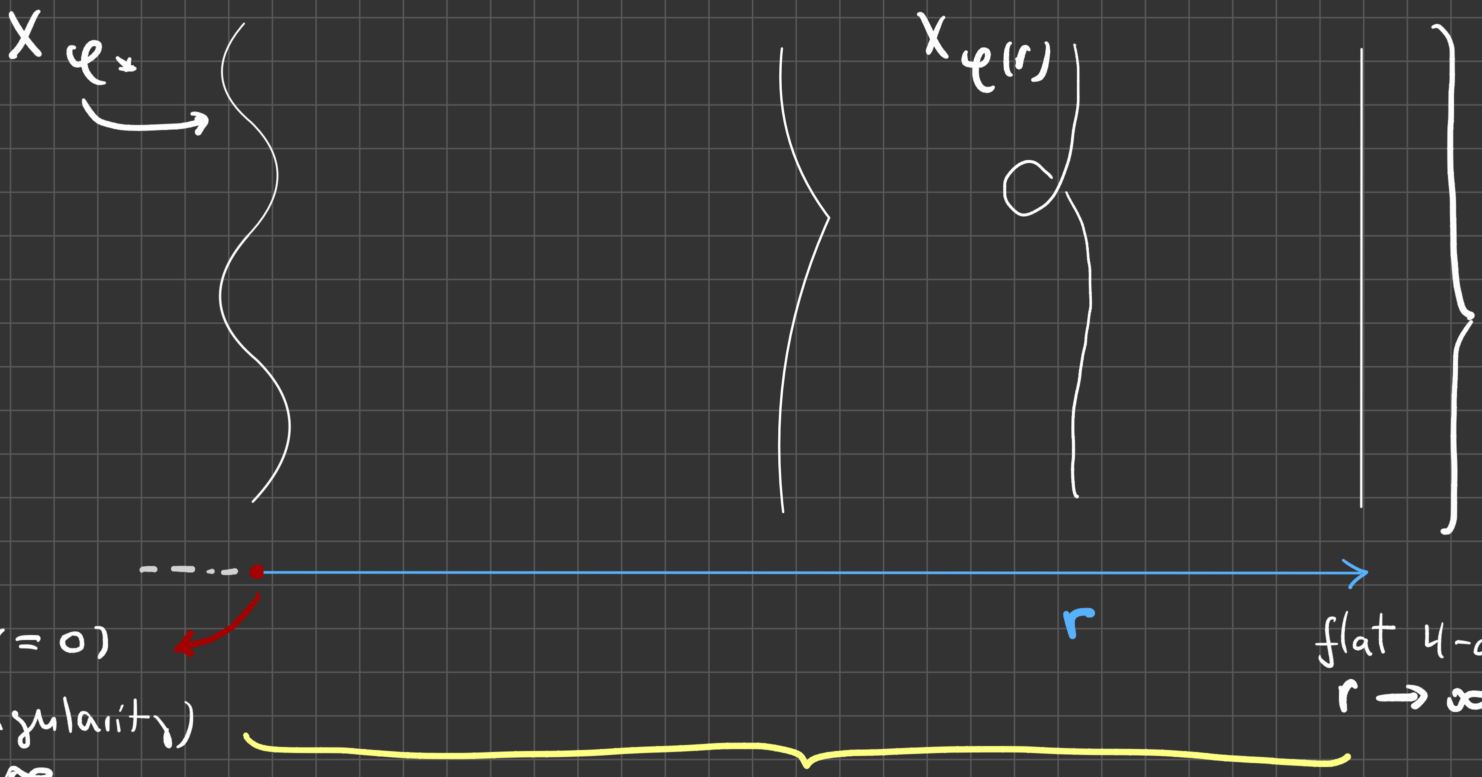
Physics : supersymmetric black hole solutions of type IIB superstrings

→ 10 dimensional

include { Einstein's equations for gravity + Maxwell equations for electromagnetism

10 dim space-time : a CY, $X_{\varphi(r)}$, at each point of 4 dim space-time

4 dim space-time = spherically symmetric asymptotically flat charged BH



CY with complex structure φ varying with r

horizon $(r=0)$

(coordinate singularity)

$u(r) \to -\infty$

- last surface visible from infinity
- surface separates exterior from interior of BH

flat 4-dim space-time

$r \to \infty \quad (u(r) \to 0)$

$$ds^2 = e^{2u(r)}\,dt^2 + e^{-2u(r)}\,d\underline{x}^2, \quad \underline{x} = (x, y, z)$$

4 dim BH metric dependent only on the radial coordinate r

- asymptotically flat : $u(r) \to 0$, $r \to \infty$
- horizon at $r = 0$: $u(r) \to -\infty$

Type IIB supergravity contains gravity but also extra $U(1)$ gauge fields (b_3 of them)

So: the BH has electric & magnetic charges

$$Q = \begin{pmatrix} q_a \\ \\ p^b \end{pmatrix} \qquad a, b = 0, 1, \ldots, h^{2,1}$$

these must all be integers

Let $\Gamma = p^a \alpha_a - q_a \beta^a \in H^3(X, \mathbb{Z})$

charge vector

(α_a, β^b) symplectic basis of $H^3(X, \mathbb{Z})$

Black hole solutions which preserve supersymmetry need to satisfy 1st order differential eqs for $U(r)$ & $\varphi(r)$, the attractor equations

These equations represent a non-linear dynamical system on the $\mathbb{C}$-structure moduli space with flow parameter $\rho = 1/r$

Let $\quad Z_\gamma = e^{K/2} \int_\gamma \Omega \quad , \quad \gamma =$ Poincaré dual of Γ.

Given a choice of $\Gamma \in H^3(X, \mathbb{Z})$ one can prove using the attractor equations that

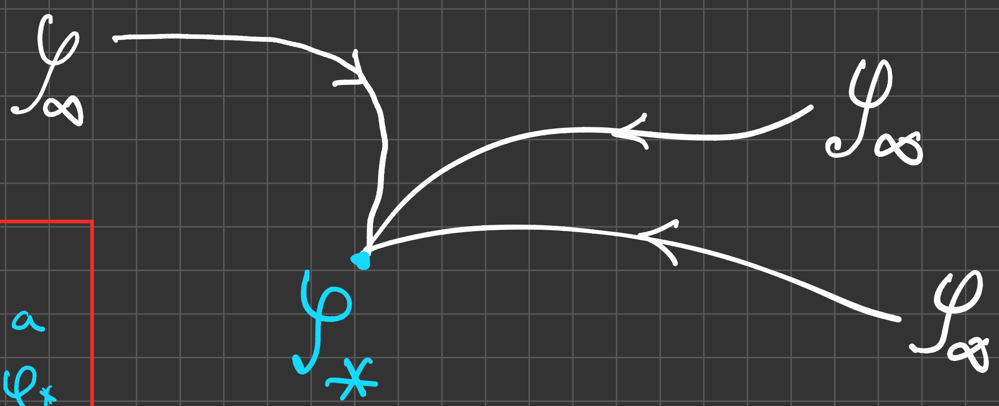
- the $\mathbb{C}$-structure parameters flow to a value $\varphi_* = \varphi(r=0)$ where $|Z_\gamma|$ reaches a minimum, and it is independent of the starting value $\varphi_\infty = \varphi(r=\infty)$

95: Ferrara + Kallosh

. . . . .

98: G. Moore
conjectures on the arithmetic nature of attractor varieties

$\hookrightarrow X_{\varphi_*} = X_*$



φ evolves smoothly to a fixed point φ_* at $r=0$

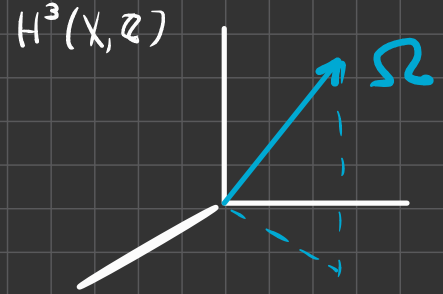
▸ Moreover,

the Q-structure at an attractor point $\varphi = \varphi_*$ is st

$$\boxed{\begin{aligned} \Gamma = p^a \alpha_a - q_a \beta^a \quad &\in \quad H^{(3,0)} \oplus H^{(0,3)} \\ \text{ie} \quad \Gamma^{(1,2)} = \Gamma^{(2,1)} &= 0 \end{aligned}}$$

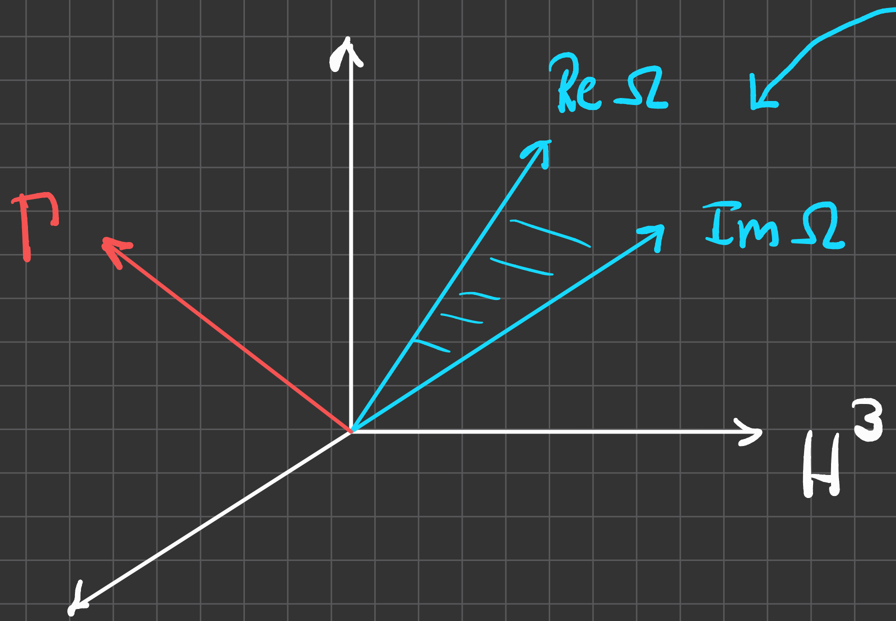
proof: exercise in special geometry

$\boxed{\text{rank 1}}$ attractors



Recall: Ω defines a line in $H^3(X,\mathbb{Z})$

Ω moves inside $H^3(X,\mathbb{Z})$ as we change the $\mathbb{C}$S



Consider

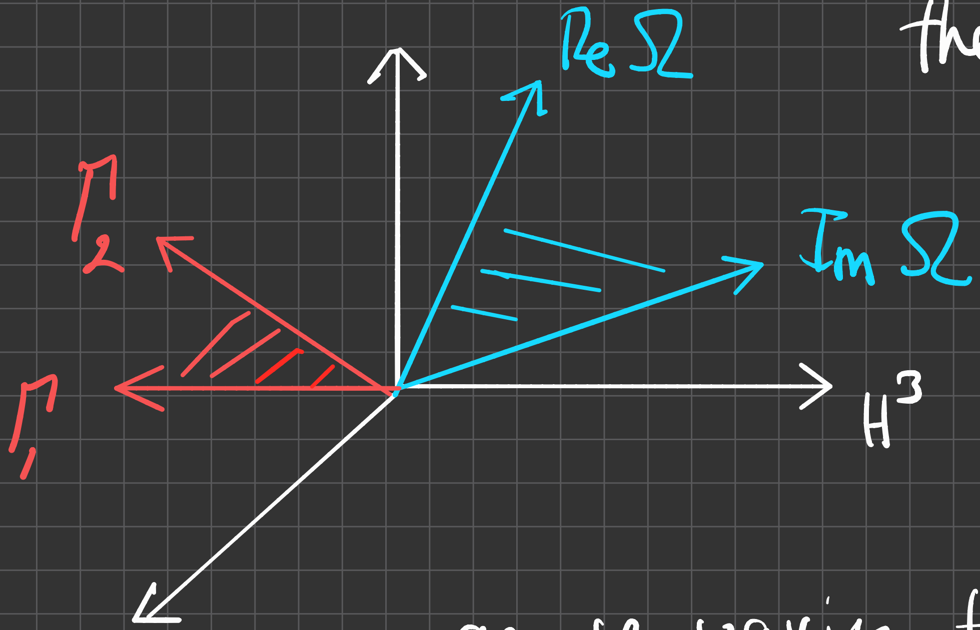
$V_{\mathbb{R}}(\varphi)$ = plane spanned over $\mathbb{R}$ by $\mathrm{Re}\,\Omega$ & $\mathrm{Im}\,\Omega$

$V_{\mathbb{R}}(\varphi)$ moves with φ

OTOH: Inside $H^3(X,\mathbb{R})$ we have a lattice of vectors $\Gamma \subset H^3(X,\mathbb{Z})$ which are fixed

rank 1 A.P: φ st $V_{\mathbb{R}}(\varphi)$ contains the line Γ

$\boxed{\text{rank } 2}$ attractors



at an attractor point of rank 2 there are $\boxed{\text{two}}$ vectors in $H^3(X,\mathbb{Z})$

Γ_1, Γ_2 st $\Gamma_{1,2} \in H^{(3,0)} \oplus H^{(0,3)}$

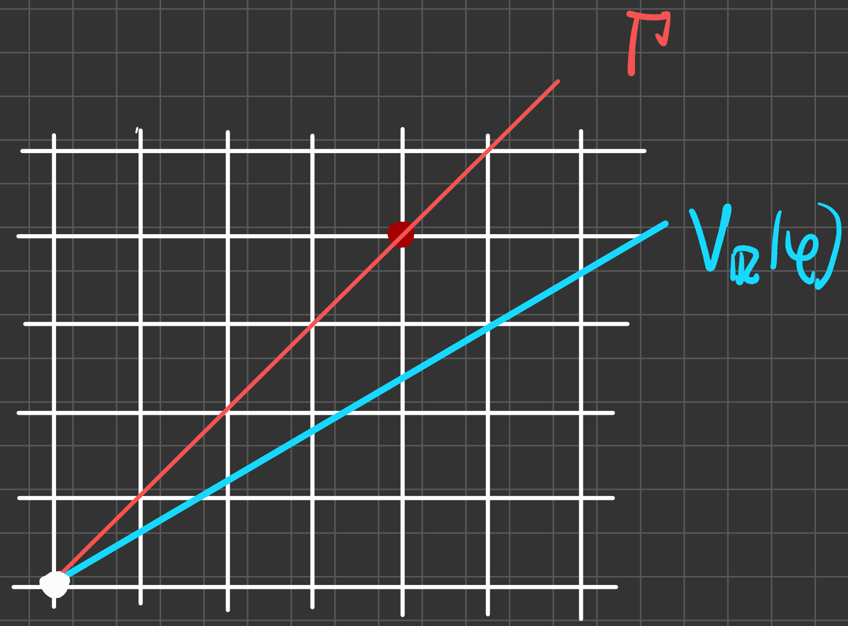
as φ varies the plane $V_{\mathbb{R}}(\varphi)$ moves and at a rank 2 attractor point $\varphi = \varphi_*$ the plane $V_{\mathbb{R}}(\varphi)$ coincides with the plane generated by Γ_1 & Γ_2

RARE, very difficult to find a CY which has rank 2 attractor points

Why?

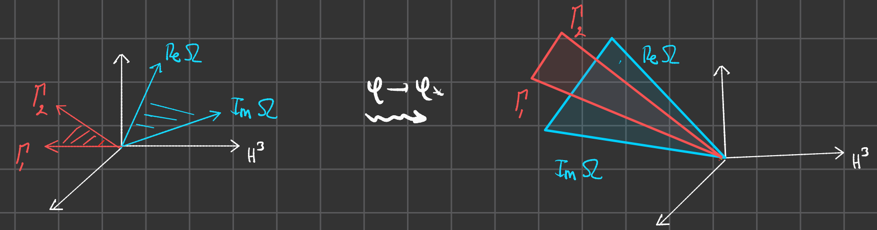
A line which passes through the origin in general will <u>not</u> pass through another lattice point unless the slope is rational.

Not too hard to find φ st $V_{\mathbb{R}}(\varphi)$ coincides with Γ



For rank 2 attractors we have a plane and it is then <u>much harder</u> to find φ st it coincides with $V_{\mathbb{R}}(\varphi)$

Geometrically: at $\varphi = \varphi^*$



$\Gamma_1, \Gamma_2 \in \Lambda := V_{\mathbb{R}}(\varphi) \cap H^3(X, \mathbb{Z})$ is a lattice plane

and $\Lambda \otimes \mathbb{C} = H^{(3,0)} \oplus H^{(0,3)}$

Also: $\Lambda^\perp \subset H^3(X, \mathbb{Z})$ lattice orthogonal to Λ

(under the natural symplectic product on 3-forms)

satisfies $\Lambda^\perp \otimes \mathbb{C} = H^{(2,1)} \oplus H^{(1,2)}$

We obtain an isomorphism

$$\boxed{H^3(X, \mathbb{Q}) = \Lambda_{\mathbb{Q}} \oplus \Lambda^\perp_{\mathbb{Q}}}$$

$\Lambda \oplus \Lambda^\perp$ has finite index extends to $\mathbb{Q}$

splitting of the Hodge structure $H^3(X, \mathbb{Q})$!

Hodge conjecture $\Rightarrow$ splitting has a geometrical origin

$\boxed{\text{VERY HARD}}$ $\rightsquigarrow$ open question

Problem: how do we find varieties with rank 2 attractor points?

However: the splitting becomes apparent in the arithmetic structure of X

$\therefore$ <u>arithmetic strategy</u> !

P. Candelas, M. Elmi
& D van Straten, 2021

P Candelas, XD, D van Straten April 2021

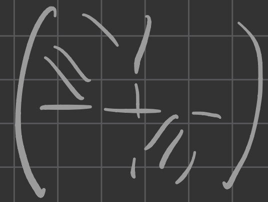
Candelas, XD, J McGovern, P Kuusela 2021 & Feb 2023

K Bönish + A Klemm + Scheidegger + Zagier 2022

P Candelas, XD, P Kuusela 2024

④ Arithmetic of attractor varieties

$\boxed{\text{At } \varphi = \varphi_*}$ Recall $H^3(X_*, \mathbb{Z}) = \underbrace{V}_{(3,0)+(0,3)} \oplus \underbrace{V^\perp}_{(2,1)+(1,2)}$

$\mathcal{U} \rightsquigarrow$ 

$$R(T) = \det(\mathbb{1} - T\,\mathcal{U})$$

$$= \underbrace{(1 - p\alpha_p T + p^3 T^2)}_{H^{2,1} \oplus H^{1,2}} \underbrace{(1 - \beta_p T + p^3 T^2)}_{H^{3,0} \oplus H^{0,3}}$$

$\boxed{\text{factors over } \mathbb{Z} \ \forall p}$

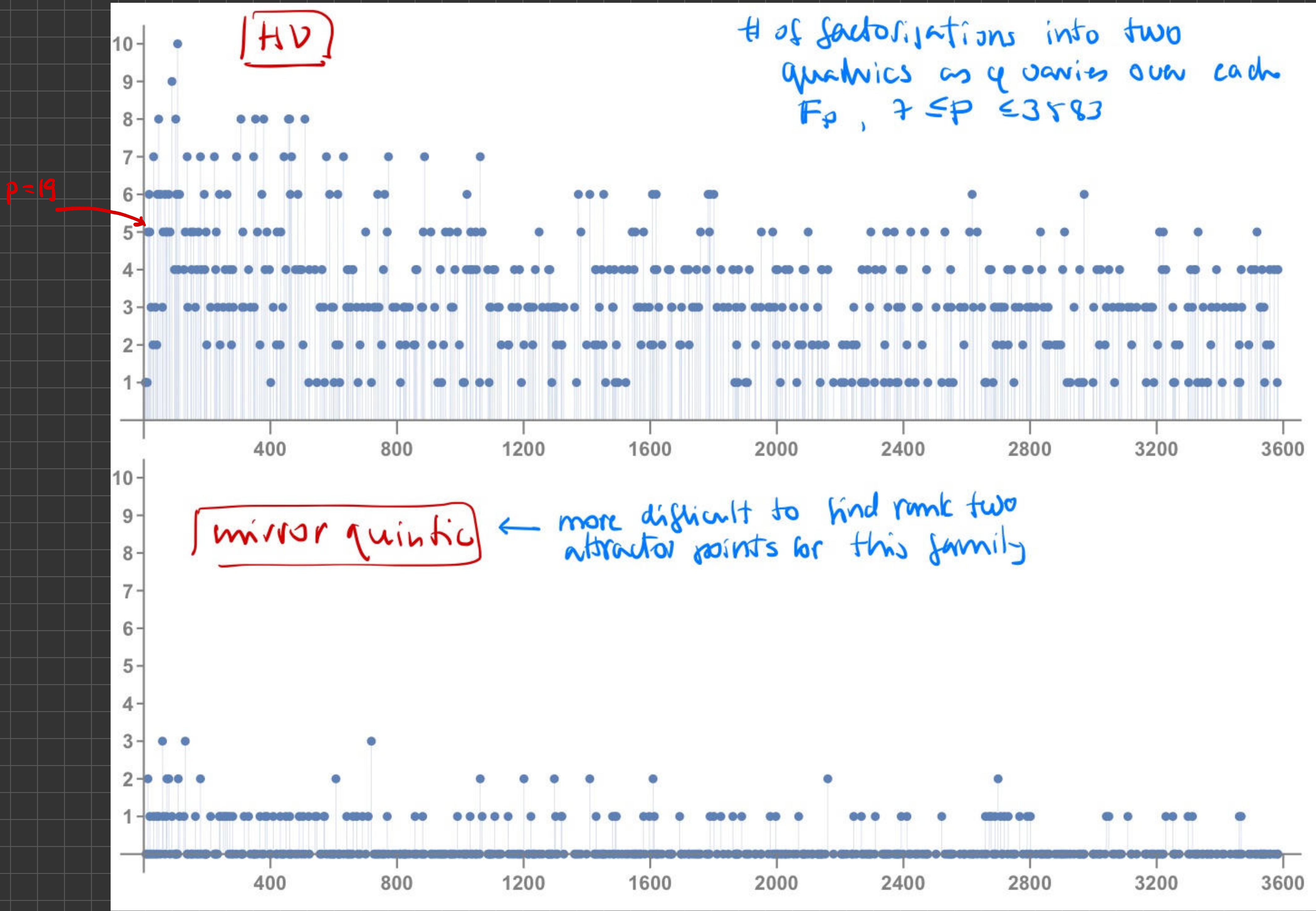
Moreover: expect α_p, β_p to be coeffs of modular forms
(Tate & Serre conjectures)

Arithmetic strategy:

make tables of $\mathcal{R}(T,\varphi)$ for many p & φ and look for persistent factorisations of $\mathcal{R}$ into two quadrics

factorisations occuring when φ_* is a root of a polynomial with integer coeffts

(P. Candelas, XD, A Thorne, Duistraten 2018
P. Candelas, XD, Duistraten 04/2021)


p=19
HV
# of factorizations into two quadrics as q varies over each $\mathbb{F}_p$, $7 \le p \le 3583$
mirror quintic
more difficult to find rank two attractor points for this family
0 1 2 3 4 5 6 7 8 9 10
400 800 1200 1600 2000 2400 2800 3200 3600

We find that for the HV manifold there is always a factorisation when (P Candelas, XD, M Elmi & D van Straten)

- $7\varphi+1=0$: $\varphi=-1/7$ $(\forall p\neq 7)$

and

- $\varphi^2-66\varphi+1=0$: $\varphi_\pm = 33\pm 8\sqrt{17}$

(exists in $\mathbb{F}_p$ when 17 is a square modulo p)

For p = 19 (say)

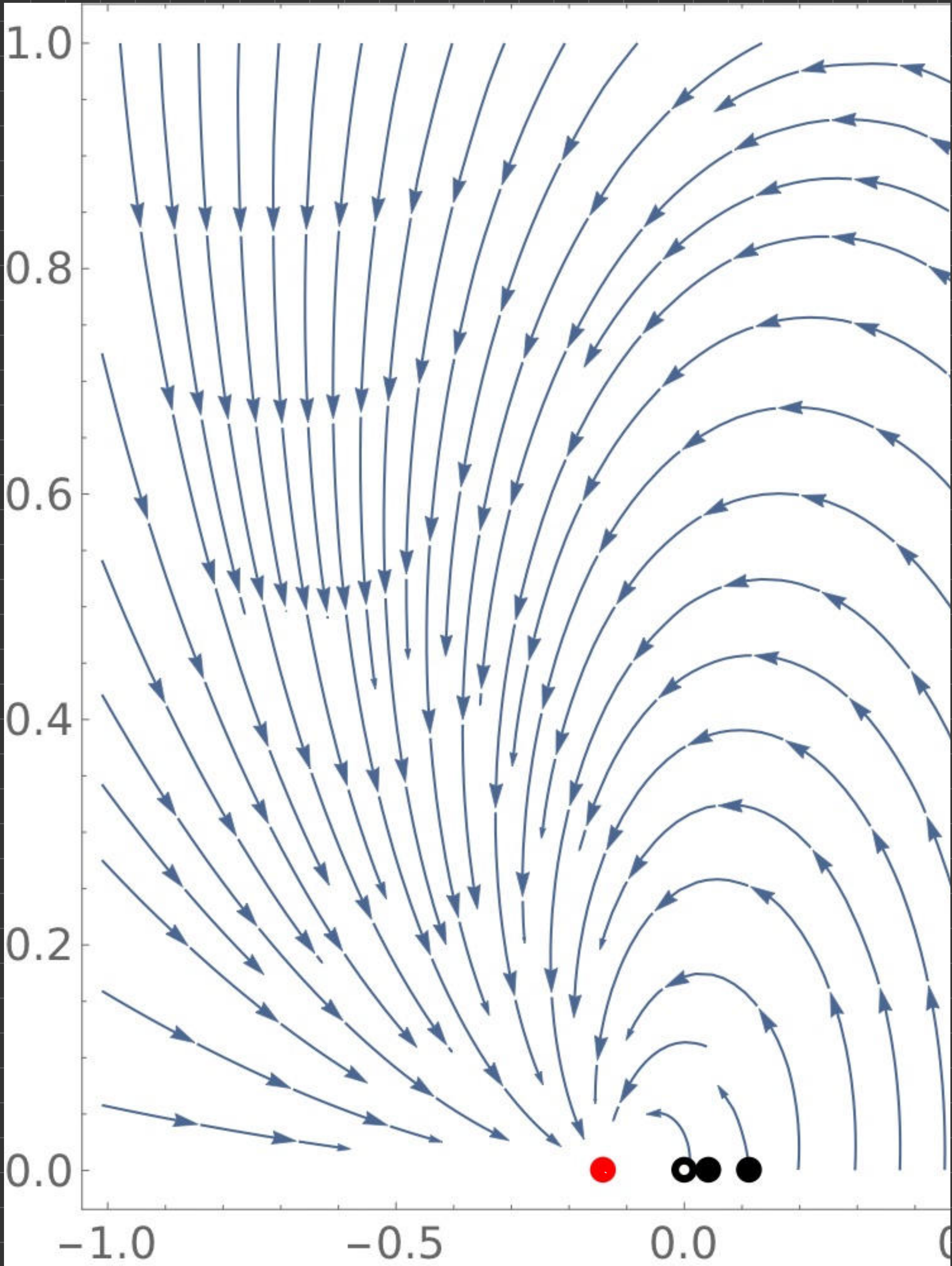
$\varphi = -\frac{1}{7} \equiv 8$, $\varphi_\pm \equiv 4, 5$

$(17 \equiv 6^2)$

	$p = 19$			
	φ	smooth/sing.	singularity	$R(T)$
conifold	1	singular	1	$(1 - pT)(1 - 20T + p^3T^2)$
	2	smooth		$1 + 4pT + 2pT^2 + 4p^4T^3 + p^6T^4$
	3	smooth		$1 - 8T + 242pT^2 - 8p^3T^3 + p^6T^4$
φ_+	4	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
φ_-	5	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
	6	smooth		$1 + 8T - 318pT^2 + 8p^3T^3 + p^6T^4$
	7	smooth		$1 - 44T - 238pT^2 - 44p^3T^3 + p^6T^4$
$\varphi = -1/7$	8	smooth		$(1 - 2pT + p^3T^2)(1 - 80T + p^3T^2)$
	9	smooth		$(1 + 4pT + p^3T^2)(1 - 160T + p^3T^2)$
	10	smooth		$1 + 12T + 562pT^2 + 12p^3T^3 + p^6T^4$
	11	smooth		$(1 + 4pT + p^3T^2)(1 - 140T + p^3T^2)$
	12	smooth		$1 + 12T + 82pT^2 + 12p^3T^3 + p^6T^4$
	13	smooth		$1 + 178T + 1082pT^2 + 178p^3T^3 + p^6T^4$
	14	smooth		$1 + 12T - 158pT^2 + 12p^3T^3 + p^6T^4$
	15	smooth		$1 + 42T - 2p^2T^2 + 42p^3T^3 + p^6T^4$
conifold	16	singular	$\frac{1}{25}$	$(1 - pT)(1 + 76T + p^3T^2)$
conifold	17	singular	$\frac{1}{9}$	$(1 - pT)(1 - 20T + p^3T^2)$
	18	smooth		$1 - 54T + 322pT^2 - 54p^3T^3 + p^6T^4$

Table 1: *The R-factors for* $\varphi \in \mathbb{F}_{19}$. *Note the factorisations into two quadrics for the five values* $\varphi = 4, 5, 8, 9, 11$.

$\varphi = -\frac{1}{7}$



There is more information in the tables:

there are modular forms

$$R(T) = (1 - p\,\alpha_p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)$$

Serre's conjecture (generalizing Taniyama-Weil)

"motives" of length two are modular

algebraically defined part of cohomology

[proof: Dieulefait, Khare & Wintenberger, Kisin]

For $\varphi = -1/7$

α_p & β_p are Fourier coefficients of a modular form for $\Gamma_0(14)$

LMFDB

$f_2 = \sum_n \alpha_n q^n$ weight 2 14.2.a.a

$f_4 = \sum_n \beta_n q^n$ weight 4 14.4.a.a

$q = e^{2\pi i \tau}$ $\tau \in \mathbb{H}$

Similarly, for $\varphi_\pm$

α_p & β_p $\longrightarrow$ coeffs of modular forms for $\Gamma_1(34) \subset \Gamma_0(34)$

$SL(2,\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod 34$

$f_2 \longrightarrow$ 34.2.b.a

$f_4 \longrightarrow$ 34.4.b.a

Area of the horizon of the BH

$\boxed{\varphi = -1/7}$

Charges $Q_{k\ell} = k(4,-15,-5,0) + \ell(0,0,2,1)$

(so a two parameter family of BHs)

let $V_* = \dfrac{7}{\pi} \dfrac{L_4(2)}{L_4(1)}$

$L_4 \to$ L-function associated to f_4

$L_4(s) = \dfrac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy\, y^{s-1} f_4(iy)$

Then

$$\boxed{A_{\ell k}(\varphi_*) = 4\pi \left| Z_\gamma(\varphi_*) \right|^2 = 14\pi \left\{ k^2 V_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{V_*} \right\}}$$

$\hookrightarrow$ BH entropy

What is the precise meaning of this?

Area of the horizon of the BH

$$\boxed{\varphi_\pm = 33 \pm 8\sqrt{17}}$$

φ_+ $\qquad Q_{kl} = k(4, -9, 7, 4) + l(4, -30, -30, -5)$

φ_- $\qquad Q_{kl} = k(-2, 0, 0, 5) + l(0, 3, 1, 0)$

$$v_* = \frac{17}{2}\left(\frac{9-\sqrt{17}}{2}\right)\frac{\lambda_4(2)}{\pi\,\lambda_4(1)} \qquad (\lambda_4 = \operatorname{Re} L_4)$$

$$A_{kl}(\varphi_\pm) = 34\pi\left(\frac{k^2}{v_*} + l^2 v_*\right)$$

Identities from mirror symmetry

$$\frac{1}{17}(9+\sqrt{17})\frac{\pi\lambda_4(1)}{\lambda_4(2)} = \frac{8}{\sqrt{15}}\left\{1-\sum_{j=1}^{\infty}\sum_{\mathfrak{p}\in\mathrm{pt}(j)} a_{\mathfrak{p}}N_{\mathfrak{p}}\left(\frac{j}{4\pi\sqrt{15}}\right)^{\ell(\mathfrak{p})}\boldsymbol{k}_{\ell(\mathfrak{p})-1}\left(\frac{\sqrt{15}}{3}\pi j\right)\right\}.$$

$$N_k = \sum_{d|k} d^3 n_d = k^3 N_k^{GW}$$

GW invariants !

Bessel function of the 3rd kind

$$k_n(z) = \sqrt{\frac{2}{\pi z}}\, K_{n+\frac{1}{2}}(z)$$

modified BF of the 2nd kind

mirror map $\quad 2\pi i t = \dfrac{\varpi_1(\varphi)}{\varpi_0(\varphi)}$

at φ_-: $\quad t(\varphi_-) = i\,\dfrac{5}{16\cdot 17}(9+\sqrt{17})\,\dfrac{\pi\lambda_4(1)}{\lambda_4(2)}$

Candelas, McGovern, Kuusela 2021 ; Candelas, XD, McGovern 2024

Outlook

- Arithmetic strategy to study the Hodge structure
  - attractor mechanism ⇝ still open questions in particular the meaning of the formula for the area of the BH solns.
  - apply to other cases! eg CM ⟷ RCFT
  - modular forms

- What makes a CY an attractor variety?

Why is the HV so special?

rigorous proof: find the geometric reason for the splitting of the Hodge structure

- IIB [X] flux vacua

Conjecture: (Kachru, Nally, Yang) A CY variety X_φ with CS φ which gives a supersymmetric flux vaccum is modular in the sense that it is associated to a modular form of $w=2$

(...... Candelas, XD, McGovern, Kuusela 2023)

Mirror symmetry

- L function of a manifold and its mirror

  (• PC, XD & FRV, 2004 • Daqing Wan, 2006
  • C Doran, T. Kelly, A. Salerno, S. Sperber, J. Voight, U. Whitcher, 2018 ×2
  • A Salerno + U Whitcher 2021 , . . .)

- Identities

  GW invariants $\longleftrightarrow$ L-values & point counting

- mirror map: $q = e^{2\pi i t}$ $\qquad t = \frac{1}{2\pi i} \frac{\varpi_1(\psi)}{\varpi_0(\psi)}$

$$\begin{aligned}
\psi^{-1} = \frac{1}{q} &+ 770 + 421375q + 274007500q^2 + 236982309375q^3 \\
&+ 251719793608904q^4 + 304471121626588125q^5 \\
&+ 401431674714748714500q^6 \\
&+ 562487442070502650877500q^7 + \cdots .
\end{aligned}$$

Krattenthaler & Rivoal (2007): proved integrality

▶ Modularity of CY varieties?

1-parameter families $\rightsquigarrow$ paramodular forms of $\Gamma(N) \subseteq Sp(4)^\vee$

$\rightsquigarrow$ at certain values of the parameter this "degenerates"
into known modular forms of $\Gamma_0(N) \subseteq SL(2,\mathbb{Z})$

THANKS!